Chapter 5.4: The Fundamental Theorem of Calculus

(the moment you have been all waiting for)

Mean Value Theorem Again

Let f be continuous on $[a,b]$. Then there exists a c in $[a,b]$ such that

$$f(c) = \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{\text{Average value of } f(x) \text{ on } [a,b]}$$

Idea: Use the Intermediate Value Theorem.
Let m be the minimum of $f(x)$ on $[a,b]$.
Let M be the maximum of $f(x)$ on $[a,b]$.

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a),$$

$$m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M$$

and so the Intermediate Value Theorem yields the existence of the desired c.

Cumulative $F(x)$

Let $f(x)$ be a continuous function on $[a, b]$. Define on $[a, b]$ a new *cumulative* function $F(x)$ as

$$F(x) = \int_a^x f(t)\, dt.$$

Relation of $f(x)$ and $F(X)$:

$$\frac{d}{dx} F(x) = f(x)$$

So $F(x)$ is an antiderivative of $f(x)$.

Fundamental Theorem of Calculus, Part I

Let $f(x)$ be a continuous function on $[a, b]$ and $F(x) = \int_a^x f(t)\, dt$. Then

$$F'(x) = \frac{d}{dx}F(x) = \frac{d}{dx}\left[\int_a^x f(t)\, dt\right] = f(x)$$

Goal:

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

$$\frac{F(x+h) - F(x)}{h} =$$
$$= \frac{1}{h}\left[\int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt\right]$$
$$= \frac{1}{h}\left[\int_a^{x+h} f(t)\, dt + \int_x^a f(t)\, dt\right]$$
$$= \frac{1}{h}\int_x^{x+h} f(t)\, dt$$

By the Mean Value Theorem, there exists $c \in [x, x+h]$

$$\frac{1}{h}\int_x^{x+h} f(t)\, dt = f(c)$$

As $h \to 0$, we have that $c \to x$ and thus

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

Examples for $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$

- $\frac{d}{dx}\left(\int_{10}^{x} \sin(t^4)e^t\,dt\right) = \sin(t^4)e^t$
- $\frac{d}{dx}\left(\int_{x}^{5} 3t\sin(t)\,dt\right) = \frac{d}{dx}\left[-\int_{5}^{x} 3t\sin(t)\,dt\right] = -3x\sin(x)$
- $\frac{d}{dx}\left(\int_{1}^{x} (t^3+1)\,dt\right) = x^3+1$
- $\frac{d}{dx}\left(\int_{x}^{5} cos(t^3)-5\,dt\right) = -cos(x^3)+5$
- $\frac{d}{dx}\left(\int_{3}^{7} t^2\,dt\right) = 0$ since we are taking a derivative of a constant
- $\int_{3}^{x} f(t)\,dt = x^2-9$ Find $f(x) = 2x$
  Take the derivative of both sides: $\frac{d}{dx}\left(\int_3^x f(t)\,dt\right) = 2x$
- $\frac{d}{dx}\left(\int_{1}^{x^2} \cos(t)\,dt\right) =$ on the next slide

Examples for $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$

- $\displaystyle \frac{d}{dx}\left(\int_1^{x^2} \cos(t)\,dt\right) =$

  Think of $F(x) = \int_1^x \cos(t)\,dt$.

  And we want $F(g(x)) = \int_1^{g(x)} \cos(t)\,dt$, where $g(x) = x^2$.

  We know $F'(x) = \cos(x)$ and then $F(g(x)) = F'(g(x)) \cdot g'(x)$ by the chain rule.

  $$\frac{d}{dx}\left(\int_1^{x^2} \cos(t)\,dt\right) = \cos(x^2)\cdot 2x$$

  $$\boxed{\frac{d}{dx}\left(\int_a^{g(x)} f(t)\,dt\right) = f(g(x))\cdot g'(x)}$$

- $\displaystyle \frac{d}{dx}\left(\int_{x^3}^{3} \sin(t)\,dt\right) = \frac{d}{dx}\left(-\int_3^{x^3} \sin(t)\,dt\right) = -\sin(x^3)\cdot 3x^2$
- $\displaystyle \frac{d}{dx}\left(\int_{x^3}^{5} \sin(t)\,dt\right) =$

  $\displaystyle \frac{d}{dx}\left(\int^{3} \sin(t)\,dt + \int^{5} \sin(t)\,dt\right) = -sin(x^3)\cdot 3x^2 + 0$

Chain Rule

$$\frac{d}{dx}\left(\int_a^{g(x)} f(t)\ dt\right) = f(g(x)) \cdot g'(x)$$

$$\begin{aligned}\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(t)\ dt\right) &= \frac{d}{dx}\left(\int_a^{h(x)} f(t)\ dt + \int_{g(x)}^{a} f(t)\ dt\right)\\ &= \frac{d}{dx}\left(\int_a^{h(x)} f(t)\ dt - \int_a^{g(x)} f(t)\ dt\right)\\ &= f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x)\end{aligned}$$

Example:

$$\frac{d}{dx}\left(\int_{3x}^{e^x} \sin(t)\ dt\right) = \sin(e^x) \cdot e^x - \sin(3x) \cdot 3$$

Fundamental Theorem of Calculus, Part II

Let $F(x)$ be *any* antiderivative of $f(x)$ on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Let $G(x) = \int_c^x f(t)\, dt$ for some $c \in [a, b]$. Notice $F(x) + C = G(x)$, where C is a constant.

$$\begin{aligned}\int_a^b f(x)\, dx &= \int_a^c f(x)\, dx + \int_c^b f(x)\, dx \\ &= \int_c^b f(x)\, dx - \int_c^a f(x)\, dx \\ &= G(b) - G(a) = (F(b) + C) - (F(a) + C) = F(b) - F(a)\end{aligned}$$

Examples for $\int_a^b f(x)\, dx = F(b) - F(a)$

$$\int_0^2 (2x+3)\, dx = \left[x^2 + 3x\right]_0^2 = (2^2 + 3\cdot 2) - (0^2 + 3\cdot 0) = 10$$

- $\int_{-2}^{0} 2t + 5\, dt = \left[t^2 + 5t\right]_{-2}^{0}$
  $= 0 - (4 - 10) = +6$
- $\int_{-2}^{3} 2xe^{x^2}\, dx = \left[e^{x^2}\right]_{-2}^{3} = e^9 - e^4$
- $\int_0^{\pi} \cos(x)\, dx = [\sin(x)]_0^{\pi}$
  $= \sin(\pi) - \sin(0) = 0$
- $\int_0^1 \frac{1}{1+z^2}\, dz = [\arctan(z)]_0^1$
  $= \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

Scary bonus: $\int_{-1}^{1} \frac{1}{x^2}\, dx = \left[-\frac{1}{x}\right]_{-1}^{1} = -1 - \left(-\frac{1}{-1}\right) = -2$ But $\frac{1}{x^2} > 0$? What?!?